GPS RECEIVERS AND RELATIVITY*

Marc Weiss
National Institute of Standards and Technology
Time and Frequency Division
325 Broadway, Boulder, CO 80303

Neil Ashby
Department of Physics, Campus Box 390
University of Colorado
Boulder, CO 80309-0390

Abstract

We illustrate the general methods for applying relativistic corrections needed by a GPS receiver in providing time or position to a user. We focus on estimating the time interval it takes for GPS signals to propagate from the transmitter to the receiver, the geometric range delay. We present a few cases which apply to many common uses of GPS. The most common case for positioning is illustrated numerically.

INTRODUCTION

A GPS receiver needs to make two corrections that are related to relativity in order to provide time or position to a user. We discuss these corrections and focus mostly on estimating the geometric range delay, Δt_D, the time for GPS signals to propagate from the transmitter to the receiver. Proper estimation of Δt_D is essential for solving for position or time. This is an application of the relativistic principle of the constancy of the velocity of light which states that electromagnetic signals travel in Euclidean straight lines with velocity *c* relative to an inertial reference frame. We present a few cases which apply to many common uses of GPS. The case where measurements of satellite signals are time-tagged at the receiver for positioning, probably the most common GPS application, is illustrated numerically.

The theory behind corrections is presented with references given for any derivations not done here. Through our theoretical discussion we show that the Interface Control Document (ICD-GPS-200) specifications, as issued by the Joint Program Office of the Global Positioning System [1], consistently cover the requirements of relativity down to the sub-nanosecond level for time. We respond to questions in the literature [2,3] as to whether the ICD specifications include relativity corrections with enough accuracy for certain applications. In particular we discuss the relativistic Doppler effect, the formula for its instantaneous magnitude, and its relationship with typical GPS receiver operation. We also address the use of carrier-phase measurements, which is not discussed in the ICD.

THEORY

Generally, a GPS navigation user measures the arrival times, on a local clock, of timing signals from at least four different satellites, then solves for four unknowns: user position x,y,z and the receiver clock offset from GPS time t. The signals from the satellites can be thought of effectively as continuous timing signals from the satellite clocks arriving at the receiver. The receiver has its own local clock for comparison. The user measures either 1) the times in the received timing signals at a specific local clock time, or 2) the arrival times on the local clock of a specific time-tag in the received timing signals. The reception time (according to the local clock) minus the transmission time at one satellite (according to GPS time) is called the *pseudorange*. These pseudoranges are used to solve for user position and time.

For the ordinary user of broadcast ephemerides, there are two and only two relativistic effects that must be considered. First, the receiver must apply a correction to the transmitted time to account for relativistic effects arising from orbit eccentricity of the transmitting satellite. This is the Δt_r term defined in the ICD. Second, the finite and universally constant speed c of signals propagating from transmitter to receiver, relative to an inertial frame (the geometric path delay), must be accounted for.

Relevant Relativity

Three effects in relativity are germane to GPS. Rates of clocks in GPS are adjusted (as for International Atomic Time) to match the rate that clocks would run on the geoid of the earth. This is a surface of gravitational equipotential in the rotating frame in which the effects 2) and 3) below add to a constant value. The relativity effects are as follows.

1) GPS time is defined using the principle of the constancy of c to synchronize an imagined system of clocks everywhere in space in the neighborhood of the earth (Einstein synchronization). GPS satellite clocks are in principle adjusted to agree with this imagined system of clocks. This network of synchronized GPS clocks realize a coordinate time. This definition of GPS time requires a locally inertial coordinate system. GPS time is thus defined relative to an earth-centered inertial coordinate system (an ECI), but the rate is set to match the rate at which clocks would run on the geoid. An ECI is also used to simplify the paths of signals propagating from satellites, since they move in Euclidean straight lines at the velocity c in vacuum relative to such inertial frames.

2) A clock moving with respect to an ECI runs slower relative to coordinate time than if it were at rest in the ECI. This is the time dilation effect due to the magnitude of the relative velocity, sometimes called the second-order Doppler effect. For satellites in GPS orbits, the fractional frequency offset needed to compensate for this is approximately $+8.3\cdot10^{-11}$ relative to the rate of clocks on the earth's geoid.

3) A clock in a lower gravitational potential runs slower relative to coordinate time than if it were at rest in a higher potential. This is called the gravitational red shift. Thus, standard clocks closer to the earth run slower than standard clocks farther away, since the potential becomes more negative closer to the earth. Clocks on GPS satellites need to be adjusted by about $-5.3\cdot10^{-10}$ relative to the earth's geoid, to compensate for this effect.

Atomic clocks in GPS satellites are given a fixed rate offset of $-4.4645\cdot10^{-10}$ as a consequence of the requirement that GPS satellite clocks run at the rate that a standard clock on the geoid would run, and of the relativistic effects in 2) and 3) for circular orbits. These three relativistic effects explain the reasons for the two corrections the user must apply. The first relativity correction is the Δt_r term defined in the ICD. This term corrects the satellite vehicle (SV) clock offset due to any eccentricity in GPS orbits. Eccentricity produces frequency offsets from the nominal fixed rate offset of $-4.4645 \cdot 10^{-10}$ due to the combined effects of 2) and 3) on the SV clock rate. The second correction applies to users' estimates of the geometric range delay, the time delay from the transmitter to the receiver if the signal traveled in vacuum. This is most easily calculated in an ECI where signals travel in Euclidean straight lines at the speed of light, the constant c.

There can be many ECI coordinate systems, differing by constant spatial rotations from each other, which serve these purposes. All frames with the same origin at the earth's center, and non-rotating with respect to the "fixed" stars, will define simultaneity in the same way. All such frames are equivalent for determining the propagation delay. Yet users need to reference their positions to the earth. Satellites broadcast their positions relative to an earth-centered, earth-fixed coordinate system (an ECEF), the WGS-84 coordinate system. Users fixed on the earth often know their coordinates as constants in the ECEF frame.

USER CORRECTIONS

At any arbitrarily chosen instant the ECEF frame coincides with an ECI frame having identical x-, y-, and z-axes, but not rotating. Removing the rotation of the earth from the ECEF defines a coordinate system that is close enough to an inertial frame to serve for estimating the path delay. As time passes the ECEF system rotates, while this ECI system remains behind, so to speak. Any such coordinate system may serve as an ECI for estimating geometric range by using the Euclidean distance between the coordinates in the ECI of the satellite at transmission and the coordinates of reception. The time for the signal to travel this path, the geometric path delay, may be estimated as the geometric range divided by the defined velocity of light, c, to better than 200 ps [4].

The principle of the constancy of the speed of light in an inertial frame requires that an ECI be used for geometric path delay if it is calculated by dividing the Euclidean distance by c. Using such an ECI greatly simplifies the problem of solving for a GPS user's position or time. Whereas GPS is intended to provide users with their position or time in the ECEF system, if we use those coordinates for geometric range with the simple Euclidean distance metric and divide by c to obtain geometric path delay, we will make significant errors.

In general, a navigation solution requires pseudo-range measurements to at least four satellites. These are used to obtain geometric range delay estimates, which in turn are used to solve for position. To use the simplicity of the Euclidean distance metric that comes with an ECI, all satellite positions at the transmission epochs must be transformed into the common ECI frame. The user may then solve for the receiver position in this coordinate frame, and for the GPS time, t, corresponding to this solution. Finally, the user must find the receiver coordinates in the ECEF at the GPS time t by accounting for the rotation of the ECEF between the chosen moment t_C at which the ECI frame is defined, and the GPS time t.

With a receiver using four satellites we generally have five different times of interest as candidates for t_C: either a single GPS transmission time and four different reception times (time tagging at transmission time), or a single reception time and four different transmission times (time tagging at reception time). In any case an ECI frame can always be found which coincides with the ECEF frame at a chosen instant of time, and in this sense the ECEF can determine an inertial coordinate system. This answers the need for a coordinate system in which the speed of light is c in vacuum during the transmission of the signals, and in which all the GPS satellite clocks can in principle be synchronized. We will give a number of examples below for using this general prescription.

Our focus here will remain on relativistic corrections for GPS users. We do not consider corrections for non-relativistic delays: ionospheric plasma delays, tropospheric delays due to water vapor, multipath interference, or receiver system delays. Non-relativistic delays must certainly be accounted for; however here our topic is relativity.

DOPPLER EFFECT

A receiver system that uses the instantaneous Doppler shift of the received carrier signal must also correct for the frequency shift of the received signal within the framework of relativity. As the true range between the satellite and receiver changes due to relative motion, the carrier frequency changes due to the Doppler effect. The relationship between the received frequency F, and the transmitted frequency, the proper frequency f, is [4,5,6,7]:

$$f = F\frac{\gamma(v)\left(1-\dfrac{N\cdot v}{c}\right)}{\gamma(V)\left(1-\dfrac{N\cdot V}{c}\right)}, \tag{1}$$

where

v = transmitter velocity in ECI coordinates,
V = receiver velocity in ECI coordinates,
N = ECI unit vector in the direction of propagation of the signal, and

$$\gamma(v)=\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}. \tag{2}$$

The change in frequency is closely related to the rate of change of range. In fact the relativistic Doppler frequency shift equation can be derived by differentiating the geometric path delay [4,6]. This proves the conceptual equivalence, within the framework of Special and General Relativity, of the methods of pseudorange (to obtain an instantaneous position solution) and integrated Doppler frequency (to obtain changes in position). A user of integrated Doppler frequency can account for the relativistic Doppler shift by accounting for the geometric path delay.

THE ROTATION MATRIX

Almost always the user of the broadcast ephemerides will want to use coordinate transformations which correspond to rotations of the coordinate system about the z-axis. For convenience we write out here an example of such a rotation matrix. Consider an ECI frame with z-axis which coincides with the WGS-84 axis. Let the position coordinates of some point of interest in these ECI coordinates be denoted by

$$X_{ECI} = \begin{bmatrix} x_{ECI} \\ y_{ECI} \\ z_{ECI} \end{bmatrix}. \tag{3}$$

Suppose that the ECI axes coincide with the ECEF axes at the time t_C. The angle through which the ECEF coordinates rotate relative to the ECI during the time interval $t\text{-}t_C$ is

$$\theta = \dot{\Omega}_e (t - t_C),$$

where:

$$\dot{\Omega}_e = 7.292115 \cdot 10^{-5} \text{ rad/s}, \tag{4}$$

the WGS84 earth rotation rate.

The time interval $t\text{-}t_C$ must be small since the earth rotation rate varies. With $t\text{-}t_C$ as large as 3 s, errors no larger than 0.05 mm are introduced [4]. With this rotation, the point denoted X_{ECI} in ECI coordinates has coordinates in the ECEF of

$$X_{ECEF} = R^S(t - t_C) X_{ECI} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{ECI} \\ y_{ECI} \\ z_{ECI} \end{bmatrix}, \tag{5}$$

where R^S is the sidereal rotation matrix. This is an example of a "passive" rotation, a rotation of the coordinate axes keeping physical position vectors unchanged.

CASES

Generally, a GPS navigation user measures the arrival times of signals from four different satellites, and uses them to solve for four unknowns: position x,y,z and time t. A common way of doing so is to make the four measurements at one instant at the receiver ("time-tagging at the receiver"). A less common method is to use signals which left the satellites at a common GPS time ("time-tagging at the transmitters"). The latter method is more complex both because the signals are not received simultaneously, and because the clocks on the satellites are synchronized for the user by adding a transmitted clock correction. The clocks themselves may differ by no more than 1 ms [1]. Thus we must account both for the motion of the user during the intervals between transmission and reception, and the differences in transmission time of a common GPS time due to clock offsets from system time.

TIME-TAGGING AT THE RECEIVER: GENERAL PRESCRIPTION

1. At a chosen reception time, measure the transmission times using the received pseudo-random noise (PRN) codes from four satellites. This gives us the time t_{SV} according to the SV clock at transmission for the signals received at the chosen reception time.

We need the GPS time of transmission for each satellite and the common reception time according to the (possibly biased) local clock. We obtain the time of reception from our local clock. The transmission time is encoded in the received signal. Since we are locked to the code, we can determine the offset of the locally generated PRN sequence required to maintain lock.

2. Apply the prescription for corrections to obtain GPS system time t of Table 20-IV, at transmission.
   This includes the eccentricity correction, Δt_r from section 20.3.3.3.3.1, an effect due to relativity. Δt_r is a correction which applies to the SV clock; it is the same correction no matter where the receiver is or how the receiver is moving. The value of Δt_r will generally be different for the clocks in different satellites.

3. Compute each satellite's position in the ECEF at its transmission time.
   This is a straightforward application of the equations of Table 20-IV for determining the x-, y-, z-coordinates of the satellite at the instant of transmission. Note that the broadcast message gives the satellite positions in ECEF coordinates--specifically in the WGS84 reference frame.

4. Choose an ECI frame for computation of the path delays.
   This choice is arbitrary, but some choices are more convenient than others. Simplification may sometimes occur if the choice is appropriately made. A natural choice for the case of time-tagging at the receiver is the GPS time equal to the local clock time of reception.

5. Transform the ECEF coordinates of each SV obtained in step 3 into the chosen ECI.
   This will normally require at least three, and perhaps four, of such ECEF position vectors to be rotated. We may freeze the ECEF at any instant to define an earth-centered inertial system. If we choose the GPS time equal to the local clock's reception time, the later corrections may be small. If the local clock is sufficiently close to GPS time at the instant of reception, then the last step below, 7, would not need to be done. Alternatively we might choose an ECI frame which matches the ECEF frame at the instant of transmission from one of the satellites.

6. Solve the path delay equations for the receiver's position and time.
   This can be done by linearizing the propagation delay equations and solving them iteratively. Note that other contributions to path delay, ionospheric and tropospheric delays, should be incorporated in this process. We use an initial estimate of position to solve for the linearized corrections to receiver position and the time offset. If these corrections are not small enough for the user, they may be used to obtain a next estimate of position and time. This, in turn, may be used to obtain the next linearized correction for position and time offset.

7. Rotate the user's position coordinates into the ECEF reference frame.
   After finding the user's position in the chosen ECI coordinate system and the correct GPS system time, the receiver's ECI position coordinates are rotated into the ECEF reference frame at the instant corresponding to the measured reception time.

TIME-TAGGING AT THE TRANSMITTER: GENERAL PRESCRIPTION

Let us now consider time-tagging at the transmitter with four satellites. This approach requires an estimate of velocity and perhaps acceleration over the interval of signal reception. If all signals leave satellites simultaneously with respect to the ECEF, most users will receive these signals within a 100 ms interval. For most users acceleration will contribute negligibly. We parallel the seven steps above.

1. Measure the time on the local clock at the reception of a given GPS time from each of four satellites.

   Each SV clock is generally offset from each other and from GPS time. We measure the time of reception of a given epoch in the code from each satellite. This, then, needs to be corrected according to the ICD by the relativistic correction Δt_r and the broadcast second-order polynomial correction for the SV clock offset from GPS time.

2. Apply the prescription for corrections to obtain GPS system time t of Table 20-IV at transmission.

   The difference here from example A is that the GPS system time t will be the same for the clocks in all satellites.

3. Compute each satellite's position in the ECEF at its transmission time.

   This is a straightforward application of the equations of Table 20-IV, for determining the x-, y-, z-coordinates of the satellite at the instant of transmission.

4. Choose an ECI frame for computation of the one-way light times.

   In this case the natural choice is to freeze the ECEF at the GPS transmission time, t. Note that the actual time of transmission will differ for each satellite, since it is determined by the SV clocks. The difference will be less than 1 ms.

5. If the ECI is defined by freezing the ECEF at GPS transmission time, the coordinates of each satellite will need to be rotated due to the time offset, Δt_{SV}, of the SV clock from GPS time. We also need to transform the estimated receiver positions at the arrival times into the ECI.

   We may use a deterministic estimate of the change in receiver position over the reception interval. Since this model is usually a velocity, we will refer to it that way here for simplicity. This velocity estimate does not change during the solution of user position and time. It is necessary to obtain the velocity estimate in the coordinates of the chosen ECI. Now we may rotate the estimated user coordinates into the chosen ECI at the first instant of reception, and use the velocity to extend to the other reception times. We assume the user clock offset from GPS time is constant over the interval of reception.

6. Solve the equations for the geometric path delays to find the receiver's position and time offset at a specific time.

   Again we linearize the propagation delay equations and solve them iteratively. We may iterate to find the position at the first reception time, using the velocity estimate to extend to the other reception times. This iteration is similar to step 6 in the of time-tagging at the receiver example.

7. Rotate the user's position coordinates into the ECEF reference frame for each of the reception times.

   If we have chosen the ECI coordinate system coinciding with the ECEF at the transmission time, we will most certainly have to perform these final rotations. For, the reception times will be at least of order 100 ms from the transmission time, and the ECEF will have changed significantly. We could perform only one rotation for the user position at the first reception time if we have the velocity vector in the ECEF as well as in the ECI. We could then use the ECEF velocity to obtain the coordinates for the other positions.

It would be possible to update the velocity estimate using positions obtained from two transmission time tags. This leads into other concerns and options associated with using GPS: filtering estimates over time. Techniques for estimating position, velocity, and acceleration could be coupled with strategies for filtering these estimates over time. We will not discuss these options here. We will mention, however, that if there are infrequent measurements of GPS signals from individual satellites, the local clock may be used to flywheel between them and find a solution. This may require careful filtering algorithms for estimation of the local clock frequency. In particular, relativistic effects on the local clock frequency due to velocity or gravitational potential may have to be considered.

GPS TIME TRANSFER WITH POSITION KNOWN

In this case we may use each pseudo-range measurement from each satellite separately to estimate our clock offset from GPS time. We either time-tag measurements at a transmission time and measure time of reception, or time-tag measurements at a reception time and determine time of transmission from the code lock of the receiver. We then use the prescriptions from Table 20-IV to obtain the satellite position in the ECEF. We must estimate the geometric path delay Δt_D in addition to the ionospheric and tropospheric corrections. Since we know the receiver position we may compute Δt_D as follows.

If ECI position vectors are referenced to the time of signal transmission, then

$$\Delta t_{GPD} = \frac{|r(t_T) - R(t_R)|}{c} \sim \frac{|r(t_T) - R(t_T)|}{c} + [R(t_T) - r(t_T)] \cdot \frac{V}{c^2}, \qquad (6)$$

where:

r is the SV position vector,
R is the receiver position vector,
t_T and t_R are GPS time at respectively transmit and receive times, and
V is the receiver velocity vector in the ECI.

In the case of an earth-fixed user,

$$V = \dot{\Omega} \times R. \qquad (7)$$

If ECI position vectors are referenced to the time of signal reception (instead of satellite transmission time), then

$$\Delta t_D = \frac{|r(t_T) - R(t_R)|}{c} \sim \frac{|r(t_R) - R(t_R)|}{c} + [R(t_R) - r(t_R)]\cdot\frac{v}{c^2}, \qquad (8)$$

where v is the satellite velocity vector in the ECI.

EXAMPLE: MULTI-CHANNEL RECEIVER, TIME-TAGGING AT THE RECEIVER

In this example we simulate a navigation solution using an example receiver position and data from the GPS constellation as it was late 1995. We choose four satellites from this constellation in view from our receiver location with elevations above 20°. Suppose the receiver is truly at ECEF latitude and longitude 35°N, 0°E, with elevation on the reference ellipsoid, and the operator desires to make a measurement of position and GPS time at t=37 240.000 000 000 0 s of the week.

The receiver ECEF coordinates at this instant will be

$$X_{WGS84} = \begin{bmatrix} R\cos 35° \\ 0 \\ R\sin 35° \end{bmatrix} = \begin{bmatrix} 5\,224\,663.389\text{ m} \\ 0 \\ 3\,658\,348.690\text{ m} \end{bmatrix}, \qquad (9)$$

where R=6378136.300 m. Such accuracy is not justified, but we are giving positions to a millimeter so that the convergence of the algorithms can be checked. The actual position is given here for comparison with the navigation solution.

We follow the step numbering from the cases section above, for this case.

1. Suppose we already have the SV clock time t_{SV} for the signals received at the chosen reception time.

2. We apply the prescription from ICD-GPS-200 according to Section 20.3.3.3.3.1 describing the user algorithm, to obtain the system time t_i for each satellite i, the GPS time of transmission. The relativistic eccentricity correction is part of this calculation. The subscript i is not used in the ICD, but is added here for clarity.

3. We obtain the ECEF coordinates of the satellites from the prescriptions given in Table 20-IV of the ICD-GPS-200. From the GPS time t_i, the time interval $t_k = t_i - t_{oe}$ from ephemeris reference epoch is calculated. Then t_k may then be used in the algorithm for computation of ephemerides. This gives the x-, y-, z-coordinates of the satellite, in the ECEF frame, at its transmission epoch.

Table I gives the results after these steps. Note that in Table I, GPS system times t_i rather than the time intervals $t_k = t_i - t_{oe}$ from ephemeris reference epoch, are used to label the events. The subscript i varying from 1 to 4 labels data from the different satellites.

Table I. GPS satellite positions in ECEF coordinates

SV#	Transmission Epoch t_i	x_i	y_i	z_i
1	37 239.924 422 365 6 s	13 005 878.255 m	18 996 947.213 m	13 246 718.721 m
2	37 239.920 713 391 8 s	20 451 225.952 m	16 359 086.310 m	-4 436 309.875 m
3	37 239.925 307 870 0 s	20 983 704.633 m	15 906 974.416 m	3 486 595.546 m
4	37 239.929 346 353 9 s	13 798 849.321 m	-8 706 113.822 m	20 959 777.407 m

4. We choose an ECI by fixing the rotation of the ECEF at t_C= 37 239.000 000 000 0 s for purposes of this example. We stress that this choice is arbitrary--much better choices are usually available. The purpose of the present choice is to illustrate its arbitrariness, and also to illustrate how rapidly the iteration algorithm converges.

5. We transform the coordinates in Table I into this ECI. The rotation matrix is different for each transmission epoch. To rotate from ECEF to ECI coordinates, the inverse of the rotation matrix of equation 5 is required. Thus we need $(R^S(t_i - t_C))^{-1}$; which must be calculated and applied to the coordinates of each satellite individually. The subscript j labels data from the different satellites after transforming to the chosen ECI. t_j will be different for each satellite, but t_C is the same for all satellites. We use a subscript C to indicate that the inertial system has been arbitrarily chosen. The results from transforming to the ECI frame are given in Table II.

Table II. Transmitted Data Transformed to the Chosen ECI System

SV #	Transmission Epoch t_i	x_j	y_j	z_j
1	37 239.924 422 365 6 s	13 004 597.642 m	18 997 823.895 m	13 246 718.721 m
2	37 239.920 713 391 8 s	20 450 127.566 m	16 360 459.358 m	-4 436 309.875 m
3	37 239.925 307 870 0 s	20 982 631.270 m	15 908 390.245 m	3 486 595.546 m
4	37 239.929 346 353 9 s	13 799 439.294 m	-8 705 178.668 m	20 959 777.407 m

6. We now solve the equations for the geometric path delays simultaneously to solving for the receiver's position and time. The notation is as follows. The four GPS satellites, at GPS times t_j, send out signals from the ECI locations $\boldsymbol{r}_j$. These four signals are received simultaneously at GPS time t by a receiver at position $\boldsymbol{R}$. The problem is to determine t and $\boldsymbol{R}$ at the receiver. The velocity of the receiver does not enter in the problem. The receiver's position $\boldsymbol{R}$ at the time of reception t will be determined by the solution of four equations which express the condition that the speed of propagation is c. The four equations to be solved are

$$(\boldsymbol{R}-\boldsymbol{r}_j)^2 - c^2(t - t_j)^2 = 0; \quad j = 1,2,3,4. \tag{10}$$

However, if the position $\boldsymbol{R}$ and time t are known approximately, the equations can be reduced to a system of four linear equations which can be solved by standard matrix inversion techniques. These equations are derived in [4], and are

$$(\boldsymbol{R}^{(i)} - \boldsymbol{r}_j)\cdot\Delta\boldsymbol{R} - c^2(t^{(i)} - t_j)\Delta t = \frac{1}{2}\left(c^2(t^{(i)} - t_j)^2 - (\boldsymbol{R}^{(i)} - \boldsymbol{r}_j)^2\right). \tag{11}$$

In equation 11, the quantity $c(t^{(i)} - t_j)$ is the i[th] estimate of the pseudorange from the receiver to the j[th] satellite, and $\boldsymbol{R}^{(i)}$ is the i[th] estimate of the receiver position in ECI coordinates. The equations (11) form a system of linear inhomogeneous equations in the corrections $\Delta\boldsymbol{R}$, Δt, which can be solved by matrix inversion. The matrix of coefficients of the unknowns $\Delta\boldsymbol{R}$, Δt will usually be nonsingular, unless the configuration of satellites is so unfavorable that the equations do not have a solution (it is possible for this to occur). The solutions obtained will be approximate, but can be used to obtain new trial values; the process of iteration can be repeated as many times as necessary to obtain the accuracy required.

To illustrate this process in the present case, as our initial guess at receiver position we take a worst case and assume the receiver is at the center of the earth. Also, we do not know the receiver clock bias so we guess that the time at the reception event is the time of the first transmission event plus some reasonable estimate of the propagation delay. In this case we set $t = t_1 + 0.075\ 00$ s. Table III gives the results at each stage of the iteration.

Table III. Results of Iterative Solution of Propagation Delay Equations

Trial #	user x-position	user y-position	user z-position	user clock time
0 (start)	0 m	0 m	0 m	37 239.999 421 454 1 s
1	5 057 363.392 m	2 355.126 m	3 541 092.792 m	37 239.997 532 305 8 s
2	5 226 931.551 m	354.224 m	3 659 938.391 m	37 240.000 033 455 9 s
3	5 224 663.780 m	380.983 m	3 658 348.973 m	37 240.000 000 006 0 s
4	5 224 663.374 m	380.988 m	3 658 348.689 m	37 240.000 000 000 0 s
5	5 224 663.374 m	380.988 m	3 658 348.689 m	37 240.000 000 000 0 s

The position converges to within 1 m after three iterations. The GPS time at reception, determined by the solution, is $t_R = 37\ 240.000\ 000\ 000\ 0$ s. Thus the receiver clock was off by exactly 1 s.

7. These results must lastly be transformed into the WGS84 system by applying the rotation $R^S(t_R - t_O)$. That is, to obtain WGS84 coordinates at the instant of reception, we must use a sidereal rotation corresponding to that instant. Upon applying this rotation to the position coordinates given in the last line of Table III, the measured values of the receiver position are

$$X_{WGS84} = \begin{bmatrix} 5\,224\,663.388 \text{ m} \\ 0 \\ 3\,658\,348.689 \text{ m} \end{bmatrix}. \qquad \textbf{(12)}$$

These agree with the true position coordinates to within a millimeter.

OTHER CONSIDERATIONS

THE BENEFIT OF BETTER INITIALIZATION

It pays to be more careful with our initial guesses. To illustrate this, let us consider the following situation which is not unreasonable after the receiver has been working for a while. There may be a good local quartz oscillator which can predict time accurately between measurements. After a local history of navigation solutions has been computed by the receiver, we may suppose that the GPS time of the next solution can be predicted to within 10 ns. If the velocity of the receiver has been monitored a reasonable estimate of the receiver position could be accurate to a few hundred meters. (An earth-fixed user can usually do much better.) Suppose we follow our example above where we assume the user chooses to take measurements at a time on the local clock exactly equal to 37 240.000 000 000 0 s, but that due to quartz oscillator instability the measurement is actually taken 10 ns later. We suppose here that the instant chosen to define the ECI frame is 37 240.000 000 000 0 s. If we go through the iterations we find two improvements. First, there is more rapid convergence, than in the first example. Better estimates of the position could reduce the number of iterations down to one. Second, the final solution for GPS time at the receiver is so close to the estimated time that the final rotation into the ECEF would not introduce significant changes in the position solution. The receiver clock can be updated with no other changes.

ERRONEOUS USE OF ECEF COORDINATES

It might seem that earth rotation effects during the propagation times from the satellites to the receiver are sufficiently small that transforming to ECI coordinates is not necessary. We tested this with the example data given in Table I, initializing the iteration as in the example but using the ECEF coordinates of the satellites. No transformation to ECI coordinates were performed. The results of the iteration, using the same algorithm as before, were significantly in error. The receiver time was in error by 14 ns and the position was in error by almost 30 m. Thus, *it is an error to solve the geometric range delay equations in ECEF coordinates.*

ITERATION WITH A SUCCESSION OF INERTIAL FRAMES

Another approach to the iteration for solving for position and time might be to consider a succession of inertial frames in the iteration process, each of which is aligned with the WGS84 frame at the reception time as estimated at the previous stage of the iteration. The advantage here is that at the end of the iteration the ECI frame would be exactly aligned with the WGS84 frame so no final rotation would be necessary. The disadvantage is the ECI coordinates of the transmission events must be recomputed that at each stage of the iteration, so more computation is required. During actual use, we might have very good estimates of the receiver position available to start the iteration. Possibly few iterations will then yield receiver position with sufficient accuracy. Then this approach might be better.

CONCLUSIONS

We have discussed the two relativity-related corrections that a GPS receiver needs to make in order to provide time or position to a user. We have explained where they come from and given general prescriptions for their application. We have illustrated these with a numerical example simulated from the actual GPS constellation. In particular we emphasize the importance of removing the rotation fromearth-fixed coordinates in order to determine the geometric range delay. This allows us to be consistent with the relativistic principle of the constancy of the velocity of light. We have also presented the equation for the relativistic Doppler shift that a receiver must account for in order to make use of instantaneous carrier phase measurements. A receiver that uses measurements of the integrated Doppler shift can correct for the geometric range in the same way as a code receiver does. Thus we conclude that the requirements if the ICD-200 which include the relativity corrections we have discussed are consistent with the requirements of relativity for range delay measurements accurate to better than 200 ps.

REFERENCES

[1] **Navstar GPS Space Segment/Navigation User Interfaces**, ICD-GPS-200, Revision C, Code Ident. No. 6Z691, ARINC Research Corporation, 4055 Hancock Street, San Diego, CA 92110 (Oct. 10, 1993).

[2] Deines, S., ``Missing Relativity Effects in GPS," paper presented at ION-GPS-90, Colorado Springs, CO, September 1990. See also ``Missing Relativity Terms in the Current GPS Algorithm," S. Deines, Contract F04701-87-C-0027, (June 1990), and ``Missing Relativity Terms in the GPS," **J. Inst. of Navigation, 39**, 111-131 (1992).

[3] Fliegel, H., and R. DiEsposti, **GPS and Relativity: an Engineering Overview**, Aerospace Report ATR-97(3389-1) (1997).

[4] Ashby, N., and Weiss, M.A., **Relativity for GPS Users**, National Institute of Standards and Technology Technical Note 1385, to be published.

[5] See Eq. (2) of "Lectures on Frequency Stability and Clocks," by R. F. C. Vessot, in **Experimental Gravitation**, ed. B. Bertotti, Academic Press, Inc., p. 155 (1974).

[6] Nelson, R., **An Analysis of General Relativity in the Global Positioning System Time Transfer Algorithm**, Sachs/Freeman Associates, Inc., Landover, Md., Contract No. N00014-90-C-204, September, 1991.

[7] Jorgenson, P.S., "Special Relativity and Intersatellite Tracking," **J. Inst. of Navigation, 35** (4), 429 (Winter 1988-1989).

Questions and Answers

BERNARD GUINOT (OBERVATOIRE de PARIS): To say this is true, of course; but I have always had some reservations about the a priori separation of velocity effect, gravitational effect, Sagnac effect and so on. The reason for this is that it is a source of many errors. I would simply mention that very often there are misunderstandings about acceleration effects; I have seen papers, especially on GPS, about these acceleration effects.

There are also many discussions about Sagnac effects. I recently received two papers from people who say that we in the time community are totally wrong because we omit the Sagnac effect due to the motion of the earth. All this happens because we do not start with the very fundamental object of general relativity, which is a metric. If you start from the metric in its usual form, you very simply or naturally get all the terms you need, all those you have mentioned here, without any discussion, without any possibility of mistakes. I think it is much better to face the problem that general relativity is some fundamental object which is metric.

MARC WEISS (NIST): Yes, I agree in principle. For most users, that brings up complexities that they do not need to worry about. I also agree that there has been confusion about accelerations. Accelerations are not a fundamental, relativistic effect; it is velocity that is the effect, and it is the fact that the tangent to the curvature is always considered to be a flat space-time that allows you to see velocity as the effect and acceleration as instantaneous effects.

The Sagnac effect is included in what I discussed in the one-way light time. Another way of understanding the Sagnac effect in GPS is simply the motion of the user relative to an inertial frame, which is what I talked about. Yes, I agree in principle, but I'm trying to show that it's a simple problem.

JOE WHITE (NRL): Marc, I am sure you can see it being simple or not, but I think it is a fairly thorough presentation of it. But, let me try to make it simpler. If people here go out and buy an off-the-shelf GPS timing receiver, is there a relativity problem they are likely to run into using it on the ground for timing?

MARC WEISS: If the receiver manufacturer has done it right, there should be no problem using it anywhere.

JOE WHITE: Are you aware of any popular receivers, at least, that do have a problem in the area?

MARC WEISS: I am not aware of them, but I have not done a great survey of what the receivers out there do. But, I would think that it would show up very rapidly. If you are sitting on the earth, even at the equator, you get 400 meters worth of error.

JOE WHITE: It shows up pretty quick.

MARC WEISS: Yes.

JOE WHITE: Now, to make it a little bit worse. Suppose we are using the receiver on an aircraft, say, a military or civilian aircraft at 40,000 feet, MACH 1-type velocities or less. Do we have an issue there?

MARC WEISS: There is no issue anywhere as long as you are doing the GPS problem; that is you have four or more satellites in view; you are trying to get GPS time; you are not using your local clock as a flywheel except for very short time periods between receiver updates, say 10-20 seconds, a minute or two; there is no problem.

JOE WHITE: Okay, that is really what I was trying to get at to kind of put this in scope for the audience to get a feel for whether there is something that we as users, as opposed to designers, might need to worry about.

MARC WEISS: The only place where another relativity term comes in is if you have very infrequent receiver updates, and now you have to use your clock as flywheel as you go through velocities and accelerations. But in that case, there are other concerns besides relativity that come in, I mean, clocks run at different rates due to shock and vibration and temperature, and you have really got a lot to keep track of, I think. You have a whole clock prediction problem in that case.